



MAXIMUM ENTROPY

Consider $S = \{s_0, s_1, s_2, s_3\}$
alphabet of size = 4

$$\begin{aligned} P_0 &= 0.5 & P_1 &= 0.25 \\ P_1(s_2) &= P_2 = P_1(s_3) \\ &= P_3 \\ &= 0.125 \end{aligned}$$

$$H(S) = 0.5 \log_2 \frac{1}{0.5} + 0.25 \log_2 \frac{1}{0.25}$$

$$+ 2 \times 0.125 \log_2 \frac{1}{0.125}$$

$$= \frac{1}{2} \times 1 + \frac{1}{4} \times 2 + 2 \cdot \frac{1}{8} \times 3$$

$$= \frac{7}{4} \text{ bits/sym} = 1.75 \text{ bits/sym}$$

Encoding

0	1 bit
10	2 bits
111	3 bits
110	3 bits

Average # bits/symbol

s_0
 s_1
 s_2
 s_3

$$\begin{aligned}
 &= 1 \times 0.5 + 2 \times 0.25 \\
 &\quad + 2 \times 3 \times 0.125 \\
 &= 1.75 = \underline{H(S)}
 \end{aligned}$$

Important Relation
 between Entropy
 & average # bits
 per symbol to encode
 source.

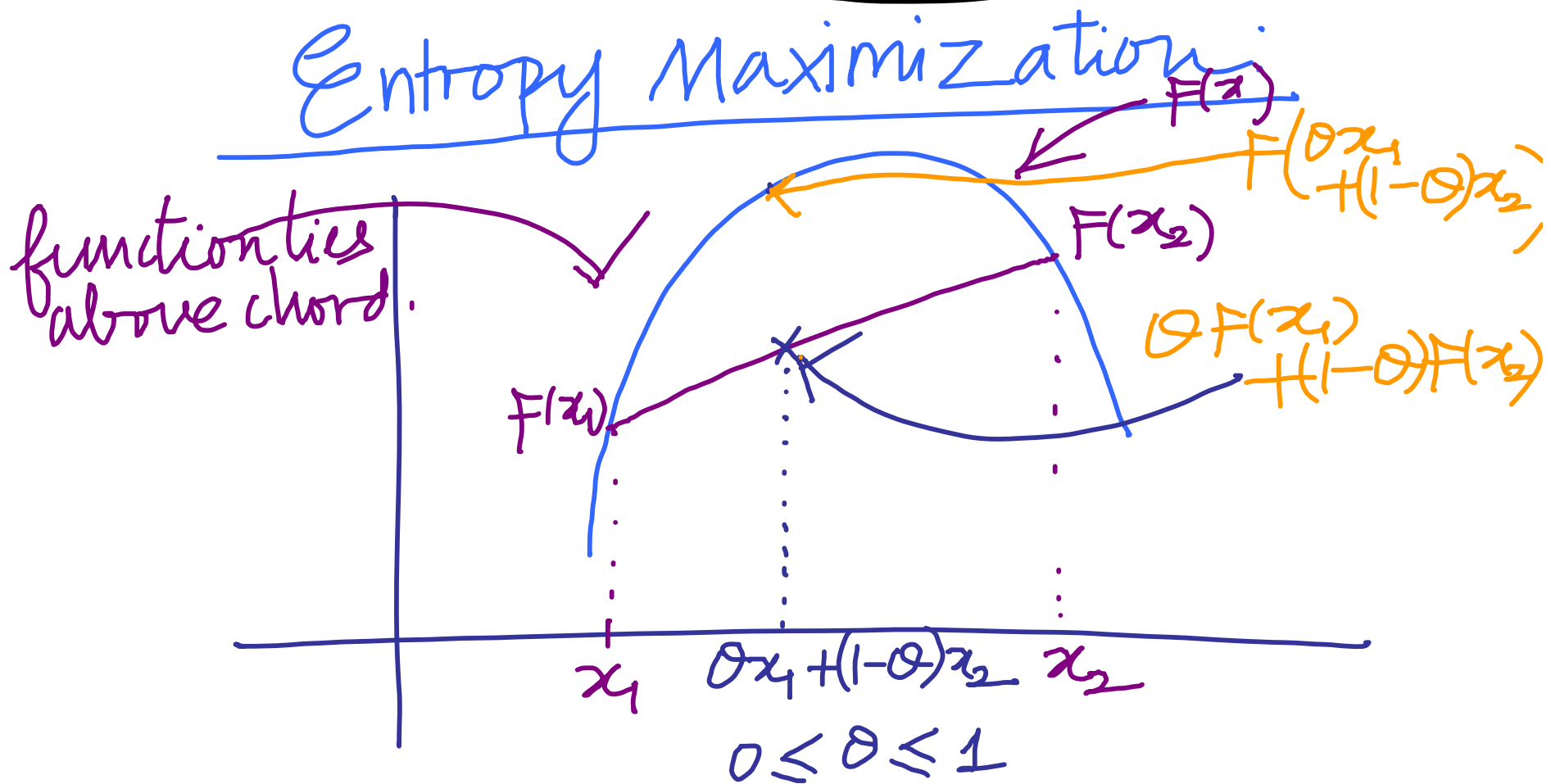
$$S = \{s_1, s_2, s_3\}$$

Each has probability
 $= \frac{1}{4}$

$$H(S) = 4 \cdot \frac{1}{4} \cdot \log_2 4$$

$$= 1 \times 2 = 2 \text{ bits}$$

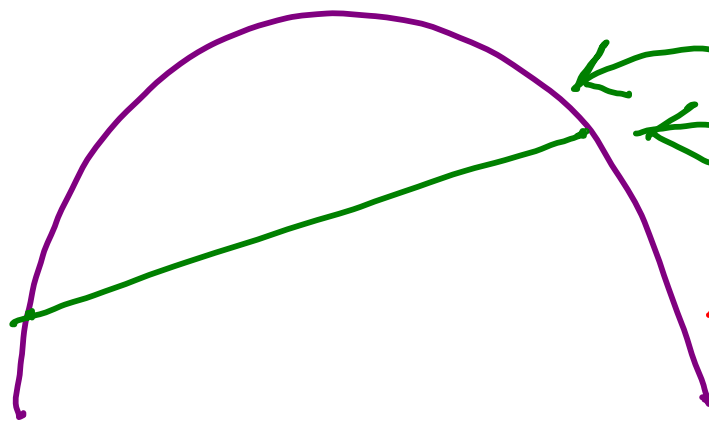
Entropy Maximization



For any x_1, x_2
 $0 \leq \theta \leq 1$

$$f(\theta x_1 + (1-\theta)x_2) \geq \theta f(x_1) + (1-\theta)f(x_2)$$

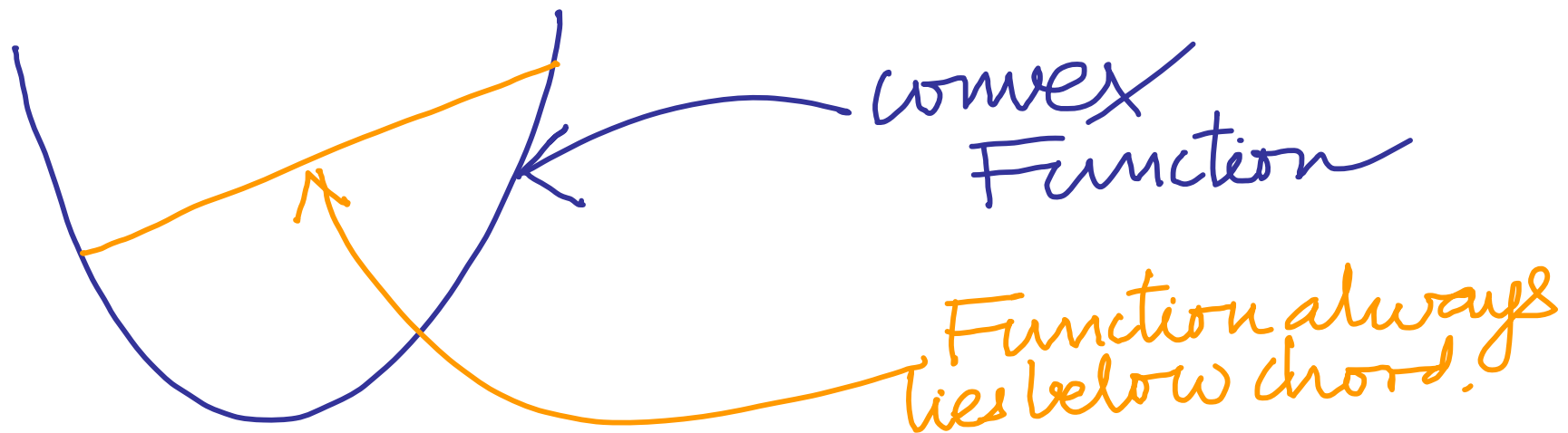
Concave
Function



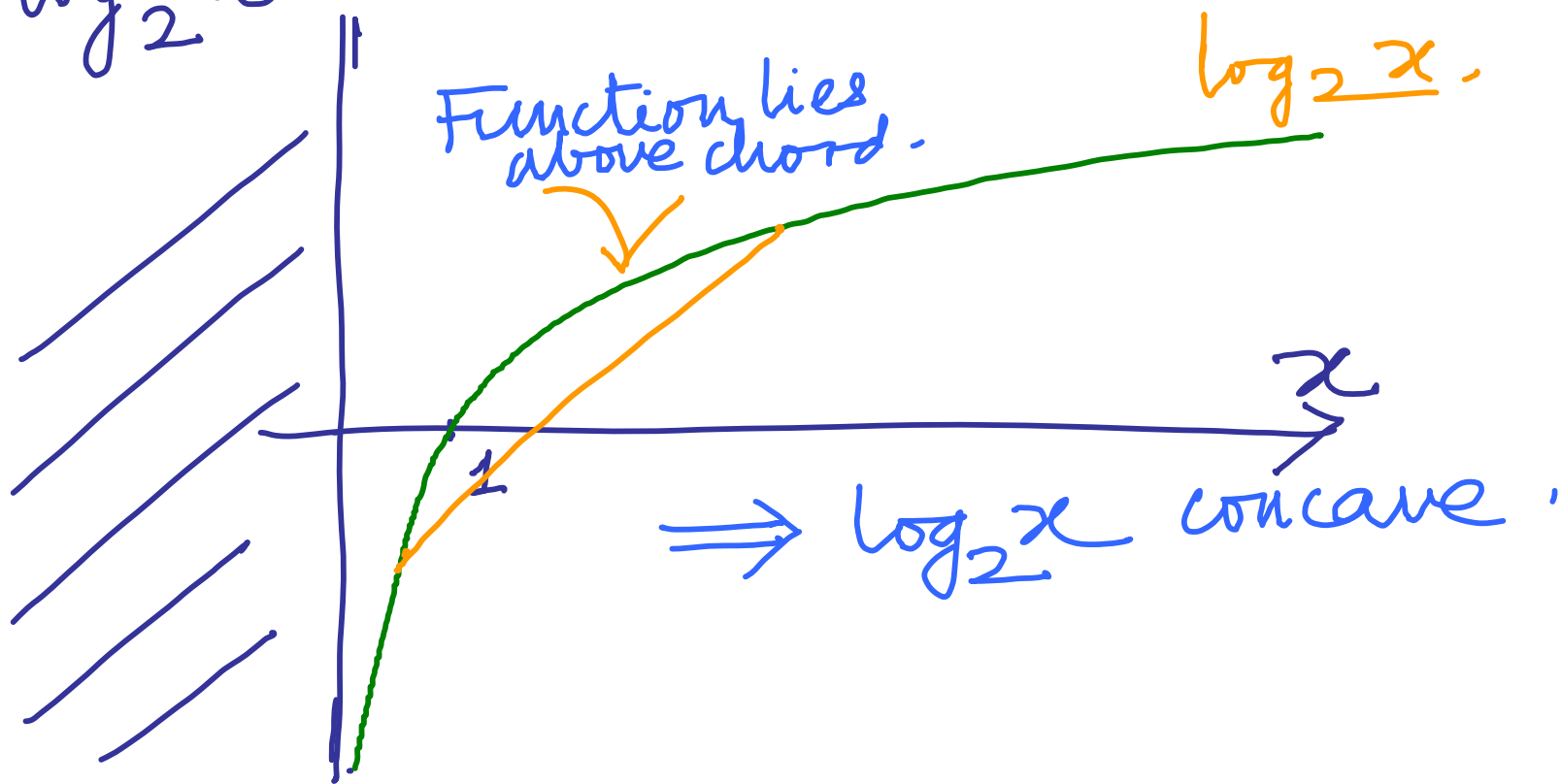
Concave
Function

Function
always lies above
chord.

Looks like an
inverted bowl.



$$f(x) = \log_2 x$$



$$f(x) = \frac{1}{\ln 2} \cdot \frac{1}{x}$$

$$f''(x) = \frac{1}{\ln 2} \cdot \left(-\frac{1}{x^2}\right)$$



$$f''(x) \leq 0$$

$$\Rightarrow f(x) = \log_2 x$$

is concave

Second Derivative Test

$$f(\theta x_1 + (1-\theta)x_2) \geq \theta f(x_1) + (1-\theta)f(x_2)$$

$$\begin{aligned} \theta &= p_1 \\ 1-\theta &= p_2 \end{aligned} \quad \begin{aligned} p_1 + p_2 &= 1 \\ \Rightarrow p_1, p_2 &\text{ are probabilities.} \end{aligned}$$

$$F(p_1 x_1 + p_2 x_2) \geq p_1 F(x_1) + p_2 F(x_2)$$

Concave Functions

$$p_0, p_1, \dots, p_{M-1}$$

$$F\left(\underbrace{\sum_{i=0}^{M-1} p_i x_i}_{E\{X\}}\right) \geq \underbrace{\sum_{i=0}^{M-1} p_i F(x_i)}_{E\{F(X)\}}$$

generalizes definition
of concavity for
M points.

$$\sum_{i=0}^{M-1} p_i x_i = E\{X\}.$$

$$F(E\{X\}) \geq E\{F(X)\}$$

Jensen's Inequality
For concave function

Inequality is reversed for a convex function

$$H(S) = \sum_{i=0}^{M-1} P_i \log_2 \left(\frac{1}{P_i} \right)$$

$\nwarrow$ $\nwarrow$ $\nwarrow$
 $\log_2$ $\frac{1}{P_i}$ x_i
 $\nwarrow$ $\nwarrow$ $\nwarrow$
 concave

$$\leq \log_2 \sum_{i=0}^{M-1} \left(P_i \cdot \frac{1}{P_i} \right)$$

$$= \log_2 \left(\sum_{i=1}^{M-1} 1 \right)$$

$$H(S) \leq \log_2 M$$

Maximum Entropy.

symbols in source alphabet

Maximum occurs when

$$\Pr(s_i) = \frac{1}{M}$$

all symbols are equiprobable

$$H(S) = M \times \frac{1}{M} \log_2 M = \log_2 M$$

Entropy is maximum when
all symbols are equiprobable.

$$P_0 = P_1 = \dots = P_{m-1} = \frac{1}{M}$$
